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Construction and exact solution of a nonlinear quantum field model in quasi-higher dimension

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Abstract

Nonperturbative exact solutions are allowed for quantum integrable models in one space-dimension. Going beyond this class we propose an alternative Lax matrix approach, exploiting the hidden multi-space–time concept in integrable systems and construct a novel nonlinear Schrödinger quantum field model in quasi-two dimensions. An intriguing field commutator is discovered, confirming the integrability of the model and yielding its exact Bethe ansatz solution with rich scattering and bound-state properties. The universality of the scheme is expected to cover diverse models, opening up a new direction in the field.

1. Introduction and motivation

A large number of quantum models in one space-dimension (1D) admits exact nonperturbative solutions, in spite of their nonlinear interaction. This exclusive class of models, which also includes field models, constitute the family of quantum integrable (QI) systems [1–5] with extraordinary properties, like association with a quantum Lax and a quantum R matrix, possessing rich underlying algebraic structures to satisfy the quantum Yang–Baxter equation (QYBE), existence of a commuting set of conserved operators with an exact solution of their eigenvalue

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problem (EVP), etc. This exact method of solution, known as the Bethe ansatz (BA), was pioneered by Bethe way back in 1931 [6] and generalized later to algebraic BA [1–4]. These QI systems defined in $(1+1)$-dimensions, include a wide variety of models, e.g. isotropic [6] and anisotropic [7,8] quantum spin-$\frac{1}{2}$ chains, δ and δ'-function Bose [9,10] and anyon [11,12] gases, nonlinear Schrödinger (NLS) field [1,13] and lattice [3] model, relativistic [14] and nonrelativistic [15] Toda chain, t–J [16] and Hubbard [17,18] model, Gaudin model [19], derivative NLS [20], sine-Gordon [21] and Liouville [22] model, etc. The algebraic structures underlying these models are also rich and diverse, which include canonical, bosonic, fermionic, anyonic and spin algebras, quantum oscillator and quantum group algebras, etc., having inherent Hopf algebra properties [23]. However, it is important to note, that among this diversity there is a deep unity, revealing that all known QI models, we are interested in, are realizable from a single ancestor Lax matrix or its q-deformation [3,24]. At the same time, the diverse algebras underlying these integrable models are also reducible from the ancestor algebra or its quantum-deformation [24]. There is a separate class of models with long-range interactions [25–27], which although are solvable quantum many body systems, exhibit different properties than those listed above and will not be discussed here. The ancestor model scheme, though a significant achievement in unifying and generating integrable models, seems to be also an apparent disappointment, since it looks like a no-go theorem, allowing no construction of new integrable models beyond the known ancestor model. Moreover, since the ancestor model and hence all QI models as its descendants, are defined in 1D, it apparently excludes any construction of integrable quantum models in higher space dimensions. 2D Kitaev models [28,29], belonging to a different class, are possibly the only exception.

Therefore for a breakthrough, we look for new ideas and observe, that the rational ancestor Lax matrix depends on the spectral parameter λ only linearly, while its q-deformation depends on its trigonometric functions [3,24]. Consequently, all quantum Lax matrices of known integrable models, since realized from the ancestor model, depend also linearly (for the rational class), or trigonometrically on λ (for q-deformed class). For going beyond the prescribed form of the ancestor model, we search for a Lax matrix with higher *scaling* (or length) dimension linked to the integrable hierarchy and for introducing extra space dimension, exploit the concept of multi-space–time [30,31] $x_n, t_n,\ n = 1, 2, 3, \ldots,$ hidden in integrable systems. For a concrete application we confine to the $n = 2$ space case and propose an alternative Lax matrix approach with λ^2 dependence, focusing on the NLS field model as an example. It is quite surprising, that though such higher order Lax matrices (with higher order poles) are well known in the context of classical integrable systems, they have never been used, as far as we know, in the construction of quantum models. Note that, taking $x = x_1,\ y = x_2,\ t = t_3$ in the NLS hierarchy, would result to the inclusion of an extra space-dimension y, apart from x and the construction of a novel quasi-$(2+1)$-dimensional NLS quantum field model, involving a scalar field $q(x, y, t)$ and its conjugate $q^\dagger$. For confirming the complete integrability of the model, one needs to show the mutual commutativity of all its conserved operators, which is guaranteed when the associated Lax matrix satisfies the QYBE. However, this task for the present Lax matrix turns out to be the most difficult one, since the commutation relations (CR) for the basic fields, known for the existing QI models fail here, due to significantly different structure of our Lax matrix and its higher λ dependence. Moreover, we can no longer seek the guidance of the ancestor algebra [24], since we have gone beyond the known ancestor model. Fortunately, we could discover intriguing algebraic relations for our basic quantum fields, which solve the required QYBE with the known rational R-matrix. Since the QYBE not only proves the integrability of a quantum model, but also gives the CR between the generator of conserved operators and the generalized creation operator, we can go ahead with the application of the algebraic BA to our quasi-2D quantum field model and solve exactly the EVP for all its conserved operators including the Hamiltonian. Many particle scattering and bound states differ considerably from the known result for the 1D NLS model. The bound states, corresponding to a complex solution for the particle momentum, are found to exhibit unusual properties with a variable stability region, dependent on the particle number, coupling constant and the average particle momentum.

2. Quantum integrable models as descendants

QI models are associated with a discretized quantum Lax matrix $U^j(\lambda)$, the operator elements of which, for ensuring the integrability of the model, must satisfy certain algebraic relations, which are expressed in a compact matrix form through the QYBE

$$R(\lambda-\mu)\, U^j(\lambda) \otimes U^j(\mu) = U^j(\mu) \otimes U^j(\lambda) R(\lambda-\mu), \tag{1}$$

at each lattice site $j = 1, 2, \ldots, N$, together with an *ultralocality* condition

$$[U^j(\lambda) \otimes U^k(\mu)] = 0, \quad j \neq k. \tag{2}$$

Individual Lax matrices, each representing a particular integrable model, differ substantially in their structure, content, nature of the basic fields and underlying algebras, whereas the quantum R-matrix, appearing in the QYBE as structure constants, remains the same for all models belonging to the same class and therefore can be of only three types: rational, trigonometric and elliptic. However, in spite of widely different Lax matrices linked to the rich variety of known QI models, they are in fact realizable from a single rational ancestor Lax matrix or its q-deformed trigonometric form [24]. We will not be concerned here with the elliptic models, which are anyway few in number. The rational ancestor Lax matrix taken in the form

$$U_{rAnc}(\lambda) = \begin{pmatrix} c_1(\lambda + s^3) + c_2, & s^- \\ s^+, & c_3(\lambda - s^3) - c_4 \end{pmatrix}, \tag{3}$$

satisfies the QYBE with the well-known rational R-matrix [1], due to its underlying generalized spin algebra

$$[s^-, s^+] = 2m^+ s^3 + m^-, \; [s^3, s^\pm] = \pm s_j^\pm \tag{4}$$

where $m^+ = c_1 c_3$, $m^- = c_2 c_3 + c_1 c_4$, with c_js as Casimir operators or constant parameters admitting zero values, and is capable of generating the known quantum integrable models of the rational class. The rational quantum $R(\lambda-\mu)$ matrix in its 4×4 matrix representation may be defined through its nontrivial elements as

$$\begin{aligned} R_{11}^{11} &= R_{22}^{22} \equiv a(\lambda-\mu) = \lambda - \mu + i\alpha, \\ R_{21}^{12} &= R_{12}^{21} \equiv b(\lambda-\mu) = \lambda - \mu, \quad R_{22}^{11} = R_{11}^{22} \equiv c = i\alpha, \end{aligned} \tag{5}$$

while the trigonometric case has q-deformed elements: $a = \sinh(\lambda - \mu + i\alpha)$, $b = \sinh(\lambda - \mu)$, $c = \sinh(i\alpha)$. The representative Lax matrices of known QI models of the rational class can be recovered from the rational ancestor model (3). We present below a few of such examples to illuminate the situation. A general form for the Lax operators, which can be realized through a bosonic representation from (3) was proposed earlier [3, Ch. VIII.4].

2.1. Generation of rational models

xxx-*spin chain* [7]: The Lax matrix may be reduced from the ancestor matrix (3) at $c_1 = c_3 = 1$, $c_2 = c_4 = 0$, giving $m^+ = 1$, $m^- = 0$, which transforms ancestor algebra (4) to the spin algebra for Pauli matrices.

Lattice NLS model [3]: The Lax matrix may be obtained from (3) at the above parameter values, by mapping spin operators through the Holstein–Primakov transformation to the bosonic operators: $[q_j, q_k^\dagger] = \delta_{jk}$.

NLS field model [1]: The Lax matrix may be recovered from its lattice version at the field limit, giving the simple familiar form

$$U_{(1)} = i \begin{pmatrix} \lambda & q \\ q^\dagger & -\lambda \end{pmatrix} \tag{6}$$

with bosonic field CR: $[q(x), q^\dagger(x')] = \delta(x - x')$.

Toda chain [15]: Lax matrix may be obtained from (3) at the parameter choice $c_1 = 1$, $c_2 = c_3 = c_4 = 0$, resulting both $m^\pm = 0$, with generators of the reduced algebra realized through canonical variables $[q_j, p_k] = \delta_{jk}$.

The rest of the QI models of the rational class, like xxx-Gaudin chain, t–J and Hubbard model, etc., can also be covered by rational ancestor model (3), employing limiting procedures, higher rank representations, fermionic realizations, etc., details of which we skip.

2.2. Trigonometric models

Similarly QI models belonging to the trigonometric class, e.g. xxz spin chain, relativistic Toda chain, sine-Gordon model, Liouville model, derivative NLS model, etc., are derivable from their representative Lax matrices, which in turn can be generated from a single trigonometric ancestor Lax matrix. This ancestor matrix is a q-deformation of (3) and satisfies the QYBE with the trigonometric R-matrix, due to its underlying generalized quantum group algebra. The details, which we omit here, can be found in [24].

3. Novel quasi-2D NLS model

Since the known quantum Lax matrices (including (6)), as discussed above, inherit their properties from the ancestor models, all of them depend on the spectral parameter λ linearly (for rational models) as in (3), or on $\sin\lambda$, $\cos\lambda$ functions (for trigonometric models) and are defined in one-dimensional space. A prominent example of rational models is the $(1+1)$-dimensional NLS field model associated with the Lax matrix (6). Therefore for going beyond the known models and introducing extra dimensions, we look into the background concept of multi-space–time dimension $\{x_n\}, \{t_n\}$, $n = 1, 2, \ldots, N$, hidden in the theory of integrable systems [30,31]. In this formulation of multi-dimension one can define multiple Lax equations of the form

$$T_{x_n} = U_{(n)} T, \quad T_{t_n} = V_{(n)} T, \; n = 1, 2, \ldots, N \tag{7}$$

(here and what follows we denote partial derivatives as subscripts, as a short-hand notation) where $T = T(\lambda, q)$ is the monodromy matrix dependent on the field $q = q(\{x_n\}, \{t_n\})$, defined in multi-space–time and the generators of the infinitesimal space–time translation $U_{(n)}(\lambda)$, $V_{(n)}(\lambda)$ are the space and time Lax operators, respectively. However, since the set of Lax equations (7) is an overdetermined system, its compatibility conditions (equality of mixed derivatives) would lead to the pairing between any two Lax matrices: $(U_{(n)}, V_{(m)})$, $(U_{(n)}, U_{(m)})$, $(V_{(n)}, V_{(m)})$, due to symmetry among the variable. Consequently, this would lead to the flatness condition among each Lax pair, generating a series of classically integrable hierarchal equations as

$$\partial_{t_m} U_{(n)} - \partial_{x_n} V_{(m)} + [U_{(n)} V_{(m)}] = 0,$$
$$\partial_{x_m} U_{(n)} - \partial_{x_n} U_{(m)} + [U_{(n)} U_{(m)}] = 0, \quad n, m = 1, 2, \ldots, N, \tag{8}$$

etc., and similarly with other pairs. A possible reduction $U_n = -V_n$ may be introduced due to the simplified condition $\partial_{t_n} U_{(n)} - \partial_{x_n} V_{(n)} = 0$. The hierarchal equations (8) represent integrable systems in $(1 + 1)$-dimensions, 2-dimensions or in quasi-higher dimensions.

We intend to use this concept of the hierarchy of multiple space–times, embedded in integrable systems, for constructing quantum integrable models in quasi-$(2 + 1)$ dimensions, restricting to the case $q(x_1, x_2, t_3)$, involving two space and one time Lax operators $(U_{(1)}, U_{(2)}, V_{(3)})$, linked to space $x = x_1$, $y = x_2$ and time $t = t_3$ variables.

3.1. Alternative Lax matrix

For a concrete application, we consider the NLS family of field models, which belongs to the rational class and choose our quantum Lax matrix as its next hierarchy:

$$U_{(2)}(\lambda) = -i \begin{pmatrix} 2\lambda^2 - q^\dagger q & 2\lambda q - i q_x \\ -2\lambda q^\dagger - i q_x^\dagger & -2\lambda^2 + q^\dagger q \end{pmatrix} \tag{9}$$

It is interesting to compare the structure of Lax matrix (9) with that of the well-known NLS model (6), to note the crucial differences, that the matrix elements of (9) depend on the spectral parameter up to λ^2 (double pole) and involve field operators $q, q^\dagger, q_x, q_x^\dagger$, defined in quasi-$(2+1)$ dimensions: (x, y, t). It needs to be mentioned, that such higher order Lax matrices like (9) appearing in the integrable hierarchy are usually taken to be independent entries, not constructed solely out of lower order Lax matrices, though there are formalisms to connect them in an involved way using classical r-matrix [32]. However no quantum extension of this method is available and it is also not clear whether the general L-operators of [3,24] can be used for this purpose. Therefore, leaving aside the question about the possibility of constructing (9) from more fundamental Lax operators, we start directly with this higher order Lax operator for constructing our quantum model.

We emphasize again that, in spite of the familiarity of Lax matrix (9) in classical integrable hierarchy, such higher-pole Lax operators have been ignored so far in the context of quantum integrable models. Note that, in the hierarchal equations we consider the pairing $(U_{(2)}, V_{(3)})$ with $x = x_1$, $y = x_2$, $t = t_3$, for constructing quasi-$(2 + 1)$-dimensional model, which can also be reached by a combination of the Lax pairs $(U_{(1)}, U_{(2)})$ in 2-dimensions with $x = x_1$, $y = x_2$ and $(U_{(1)}, V_{(3)})$ in $(1 + 1)$-dimensions with $x = x_1$, $t = t_3$. Note that, the x-shift space Lax operator here can be given by $U_{(1)}$ in (6), associated with the space Lax operator of the standard NLS model, while the y-shift space Lax operator is given by $U_{(2)}$ in (9) and the t-shift time Lax operator by $V_{(3)}$, representing a higher order λ^3 dependence (cubic pole) form, which we do not reproduce here. However, for constructing our quantum model and exactly solving it through algebraic Bethe ansatz we would need only the quantum Lax operator $U_{(2)}$.

We introduce here the notion of scaling or length dimension, which is a useful concept in analyzing higher order Lax operators in multi-space–time dimensions. One defines a scaling dimension $[L^{-1}] = 1$ for length L. Therefore from (7) we get $[U_{(n)}] = n$, since $[\partial_{x_n}] = n$ (similarly for $V_{(n)}$) and consequently $[U_{(1)}] = 1$, since each term in (6) has scaling dimension $[\lambda] = [q] = 1$. Similarly, $[U_{(2)}] = 2$, since in (9) $[\lambda^2] = [|q|^2] = [q_x] = 2$, and $[V_{(3)}] = 3$, etc.

3.2. *Quantum integrability through Yang–Baxter equation*

In dealing with quantum field models one has to lattice regularize the Lax operators first to avoid short-distance singularities [1]. Therefore, our intention is to show, that the discretized Lax matrix along the y-direction: $U^j = I + \Delta U_{(2)}(\lambda, q_j)$, where $q_j = q(x, y = j, t)$, with lattice constant $\Delta \to 0$, does satisfy the QYBE (1) with the rational R-matrix (5). However, this becomes a highly involved problem, since due to more complicated structure of the present Lax matrix (9), *ten* out of total 16 relations of the 4×4 matrix QYBE remain nontrivial, all of which are to be satisfied with a suitable field CR. Compare this situation with the known 1D NLS case [1,13], where due to much simpler form of the Lax matrix $U_{(1)}$ (6), only *two* nontrivial relations in the QYBE survive, which can be solved successfully using the bosonic field CR. However, we realize that, no algebraic relations, including the bosonic CR, appearing in the existing integrable models would work here, since the choice of $U_{(2)}$ has taken us beyond the scope of the known ancestor models and the associated algebras. Moreover, the CRs for the field now have to be sought for along the extra direction y, that has been included in the system. Therefore we look for some innovative algebraic relations for the basic quantum fields to be consistent with the QYBE, linked to the present Lax matrix (9). Fortunately, we find a new set of such relations for our quasi-2D fields as

$$[q(x,y,t),\, q_x^\dagger(x,y',t)] = -2i\alpha\,\delta(y-y'),$$

$$[q_x(x,y,t),\, q^\dagger(x,y',t)] = 2i\alpha\delta(y-y'), \tag{10}$$

$$[q(x,y,t),\, q^\dagger(x,y',t)] = 0 \tag{11}$$

(or their discretized version (23)) together with their Hermitian conjugates. Note that CRs (10) (or (23)), exhibiting an asymmetry in space variables are fundamentally new relations, different from known relations like canonical, bosonic, etc. It may be observed that, the form of CR (10) may be linked to the quadratic *space-velocity* term $:q_x^\dagger q_x:$ appearing in Hamiltonian (13). Application of these algebraic relations (10)–(11) satisfies miraculously all ten nontrivial equations appearing in the QYBE, involving the discretized Lax matrix $U^j(\Delta)$, up to order $O(\Delta)$ (see (24)–(25) in Appendix A for details). This is however enough for proving the integrability of the field models, obtained at the limit $\Delta \to 0$. It is remarkable, that in spite of the presence of an x-derivative term, new CRs (10) satisfy the necessary ultralocality condition (2). This is because not x but y is the relevant space direction here, where the fields commute at space separated points along $j \to y$, reflecting the quasi-2D nature of our model.

Therefore, since due to (10)–(11) (or its discretized version (23)), the lattice regularized quantum Lax operator $U^j(\lambda)$ constructed from (9) satisfies the QYBE (1) for the rational R-matrix, together with the ultralocality condition (2), the transition matrix for our model, defined for N-lattice sites: $T(\lambda) = \prod_{j=1}^N U^j(\lambda)$, must also satisfy the QYBE [1]

$$R(\lambda-\mu)\, T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda-\mu),\ T(\lambda) = \begin{pmatrix} A(\lambda), & B(\lambda) \\ B^\dagger(\lambda), & A^\dagger(\lambda) \end{pmatrix}, \tag{12}$$

with the same $R(\lambda-\mu)$-matrix. This happens due to the coproduct property of the underlying Hopf algebra [23], which keeps an algebra invariant under its tensor product. This global QYBE (12) serves two important purposes. First, it proves the quantum integrability of the model by showing the mutual commutativity of all conserved operators. Second, it derives the commutation relations between the operator elements of $T(\lambda)$, which are used for the exact algebraic Bethe ansatz solution of the EVP.

In more details: multiplying QYBE (12) from left by R^{-1}, taking the trace from both sides and using the property of cyclic rotation of matrices under the trace, one can show that $\tau(\lambda) = \mathrm{trace}\, T(\lambda)$ commutes: $[\tau(\lambda), \tau(\mu)] = 0$. This in turn leads to the Liouville integrability condition: $[C_j, C_k] = 0,\ j,k = 1,2,\cdots$, since the conserved set of local operators are generated from $\ln \tau(\lambda) = \sum_j C_j \lambda^{-j}$, through expansion in the spectral parameter λ. Following this construction and exploiting the explicit form of the Lax matrix (9), we can derive, in principle, all conserved operators $C_j,\ j = 1,2,\ldots$ for our model. Skipping the details, which can be found for the classical case in [31], we present here only the x-shift Hamiltonian as $H_{(x)} \equiv C_2$:

$$H_{(x)} = \int dy : (iq^\dagger q_y + q_x^\dagger q_x + {q^\dagger}^2 q^2) : \tag{13}$$

and the t-shift Hamiltonian as $H \equiv C_4$:

$$H = \int dy : (iq_x^\dagger q_{xy} + q_y^\dagger q_y + i(q^\dagger q)(q^\dagger q_y - q_y^\dagger q) \\ - 2(q^\dagger q)\, q_x^\dagger q_x + {q^\dagger}^2 q_x^2 + {q_x^\dagger}^2 q^2) :, \tag{14}$$

which we take as our model Hamiltonian, we are interested in. Notice the quasi-$(2+1)$-dimensional nature of Hamiltonian (14), since though it involves both x and y derivatives of the field: $q_x(x,y,t)$ and $q_y(x,y,t)$, the volume integral is taken only along y. Asymmetry in the appearance of space derivatives is also explicit. However, at the same time an operator with double space volume integral: $\mathrm{H} = \frac{1}{L}\int_{-L}^{L} dx\, H$ in a strip of $x \in [-L, L]$ is also conserved in time, since $\partial_t \mathrm{H} = 0$.

4. Algebraic Bethe ansatz for the eigenvalue problem

Since $T(\lambda)$ satisfies the QYBE (12) with the rational R-matrix, we can follow the procedure for the algebraic BA, close to the formulation for the 1D quantum NLS model [1,13]. As we have discussed above, $\tau(\lambda) = \mathrm{trace}\, T(\lambda) = A(\lambda) + A^\dagger(\lambda)$ is linked to the generator of the conserved operators $C_j,\ j = 1,2,\ldots$, including the Hamiltonian (14). The off-diagonal elements of $T(\lambda)$ (12): $B(\lambda)$ and $B^\dagger(\lambda)$, on the other hand, can be considered as generalized *creation* and *annihilation* operators, respectively. For solving the EVP for all conserved operators: $C_j|M\rangle = c_j^{(M)}|M\rangle,\ j = 1,2,\ldots$ simultaneously, we construct exact M-particle Bethe state $|M\rangle = B(\mu_1)B(\mu_2)\cdots B(\mu_M)|0\rangle$, on a pseudo-vacuum $|0\rangle$ with the property $B^\dagger(\mu_j)|0\rangle = 0$, $A(\lambda)|0\rangle = a_0(\lambda)|0\rangle$, and aim to solve the EVP: $\tau(\lambda)|M\rangle = \Lambda_M(\lambda, \mu_1, \mu_2, \ldots, \mu_M)|M\rangle$, with exact eigenvalues $\ln \Lambda_M(\lambda, \{\mu_a\}) = \sum_j c_j^{(M)}(\{\mu_a\})\lambda^{-j}$.

4.1. Exact solution for quasi-2D quantum field model

For obtaining the final result for our quantum NLS field model on infinite space interval we follow the formulation of [1] for the 1D NLS field model, though adopted here for the transverse dimension y and higher order Lax operator (9). We switch over to the field limit: $\Delta \to 0$ with total lattice site $N \to \infty$ and then take the interval $L = N\Delta \to \infty$, assuming vanishing of the field $q_j \to 0$, at $j \to \infty$, compatible with the natural boundary condition of having the vacuum state at space infinities, yielding the asymptotic Lax matrix $U^j(\lambda)|_{j\to\infty} = U_0(\lambda) = 2i\lambda^2\sigma^3$. Therefore, we have to shift over to the field transition matrix defined as

$$T_f(\lambda) = U_0^{-N}\, T(\lambda)\, U_0^{-N}, \quad N \to \infty, \tag{15}$$

and for further construction introduce $V(\lambda,\mu) \equiv U_0(\lambda) \otimes U_0(\mu)$, $W(\lambda,\mu) = (U^j(\lambda) \otimes U^j(\mu))_{j\to\infty}$. We may check from the QYBE (1) that W satisfies the relation $R(\lambda-\mu)W(\lambda,\mu) = W(\mu,\lambda)R(\lambda-\mu)$, using which we can derive from QYBE (12), that the field transition matrix (15) also satisfies the QYBE

$$R_0(\lambda,\mu)\, T_f(\lambda) \otimes T_f(\mu) = T_f(\mu) \otimes T_f(\lambda) R_0(\lambda,\mu), \tag{16}$$

but with a transformed R-matrix:

$$R_0 = S(\mu,\lambda)R(\lambda-\mu)S(\lambda,\mu),\ S(\lambda,\mu) = W^{-N}(\lambda,\mu)V^N(\lambda,\mu),\ N \to \infty, \tag{17}$$

where $R(\lambda-\mu)$ is the original rational R-matrix (5) (see [1] for similar details on 1D NLS model).

Based on the above formulation and using the field operator products: $q_j q^\dagger_{j,x} = -2i\frac{\alpha}{\Delta}$, $q^\dagger_{j,x} q_j = 0$, at $j \to \infty$, compatible with the field CR, we can calculate explicitly the relevant objects needed for our field model. In particular, the central 2×2 block W_c for matrix W turns out to be

$$W_c(\lambda,\mu) = I + \Delta\, M(\lambda,\mu) \begin{pmatrix} (\lambda-\mu) & 0 \\ -2\alpha & -(\lambda-\mu) \end{pmatrix}, \tag{18}$$

with an intriguing factorization of its spectral dependence by a prefactor $M(\lambda,\mu) = 2(\lambda+\mu)$, which is the key reason behind the success of the exact algebraic Bethe ansatz solution for our field model, in spite of the more complicated form of its Lax operator. Note, that since our model shares the same rational R-matrix with the known NLS case (though having different Lax operators), the present result coincides in part with that of the 1D NLS model [1,13], though only formally. On the other hand, the transformed R_0 matrix, relevant for the field model, depends on the corresponding asymptotic Lax matrix and its product through matrix $S(\lambda,\mu)$. Therefore, since Lax matrix (9) for our model is more complicated, compared to (6) for the 1D NLS model, our final result shows intriguing differences from the known NLS result, which we highlight below.

For constructing R_0 using definition (17), we have to construction first matrix $S(\lambda,\mu)$, taking proper limit of W^{-N} at $L \to \infty$ using (18). Through some algebraic manipulations, which are skipped here, we finally arrive at the field limit to a simple expression for R_0 matrix, expressed through its nontrivial elements as

$$\begin{aligned} &R^{11}_{11} = R^{22}_{22} = a(\lambda-\mu),\ R^{12}_{21} = b(\lambda-\mu),\ R^{11}_{22} = R^{22}_{11} = 0, \\ &R^{21}_{12} = b(\lambda-\mu) - \frac{\alpha^2}{\lambda-\mu} + \frac{\alpha^2\pi}{M(\lambda,\mu)}\delta(\lambda-\mu), \end{aligned} \tag{19}$$

where $M(\lambda,\mu) = 2(\lambda+\mu)$, the terms $a(\lambda-\mu)$, $b(\lambda-\mu)$ are as in (5) and the $\delta(\lambda-\mu)$ term vanishes at $\lambda \neq \mu$. It is interesting to compare R_0-matrix (19), relevant for the field models, with the original R-matrix (5). Now from QYBE (16) linked to field models, we can derive for our model the required CR between the operator elements of T_f, using R_0 matrix (19). In particular, we get the commutation relation

$$A(\lambda)B(\mu_j) = (f_j(\lambda - \mu_j) + \frac{\alpha^2\pi}{M(\lambda, \mu_j)}\delta(\lambda - \mu_j))B(\mu_j)A(\lambda), \tag{20}$$

where $f_j = \frac{\lambda-\mu_j-i\alpha}{\lambda-\mu_j}$. Note that the prefactor $M^{-1}(\lambda, \mu_j) = \frac{1}{2(\lambda+\mu_j)}$ appearing in the above CR bears the imprint of the λ^2 dependence of our Lax matrix and is absent in such relations in the standard NLS model. At $\lambda \neq \mu_j$ however, when the singular term vanishes, the relation coincides formally with the known NLS case. Using this result and the property of the vacuum state $|0\rangle$, we obtain the exact EVP for

$$A(\lambda)|M\rangle = F_M|M\rangle, \text{ as } F_M = \prod_j^M f_j(\lambda - \mu_j)$$

and hence for $\tau(\lambda)$, which gives finally the exact eigenvalues for all conserved operators. For our model Hamiltonian $H = C_4$, we obtain the exact energy spectrum $E_M = \sum_j^M \mu_j^4$, for the M-particle scattering state, which clearly differs from that of the known NLS model [1,3], though bearing formal similarity with the next NLS hierarchy.

It is remarkable, that in spite of the highly nonlinear field interactions present in the Hamiltonian (14), the scattering spectrum shows no coupling between individual quasi-particles, mimicking a free-particle like scenario. On the other hand, the bound-state or the quantum *soliton* state, which is obtained for the complex *string* solution for the particle momentum: $\mu_j^{(s)} = \mu_0 + i\frac{\alpha}{2}((M+1) - j)$, where μ_0 is the average particle momentum and α is the coupling constant, induces mutual interaction between the particles. The corresponding energy spectrum may be given by

$$E_M^{(bound)} = Re[\sum_j^M (\mu_j^{(s)})^4] = M\mu_0^4 + E_b(\mu_0, \alpha, M), \tag{21}$$

where E_b is the binding energy of the $M > 1$-particle bound-state. Recall, that a bound-state becomes stable, when its energy is lower than the sum of the individual free-particle energies, which in turn is ensured by the negative values of the binding energy: E_b. More negative binding energy indicates more stable bound-states. For the known NLS model the binding energy [1,13] $E_b^{nls}(\alpha, M) = -\frac{\alpha^2}{12}M(M^2 - 1)$ is independent of μ_0 and strictly negative, which makes the corresponding bound-states always stable with the stability increasing as the particle number M and the coupling constant α increase.

However, for the present quasi-2D NLS model, the picture differs significantly, producing a fascinating bound-state spectrum with intricate stability region. Note, that in the present case the binding energy

$$E_b(\mu_0, \alpha, M) = E^+ + E^-, E^- = -\frac{\alpha^2\mu_0^2}{2}M(M^2 - 1)$$
$$E^+ = \frac{\alpha^4}{16}M[(\frac{1}{5}M^4 + 1 - \frac{2}{3}M(M + \frac{4}{5}))], \tag{22}$$

contains both negative and positive terms (due to the simple mathematical fact, that in the expression $\sum_j^M(\mu_j^{(s)})^4$ the real term with $i^2 = -1$ gives negative, while $i^4 = +1$ strictly positive contribution). Therefore, binding energy (22) may take negative as well as positive values, depending on the parameters α, μ_0 and M, defining a variable domain for the existence of stable bound states (see Fig. 1). Note that the term $E^- = \frac{\mu_0^2}{6}E_b^{nls}$, proportional to that of the known

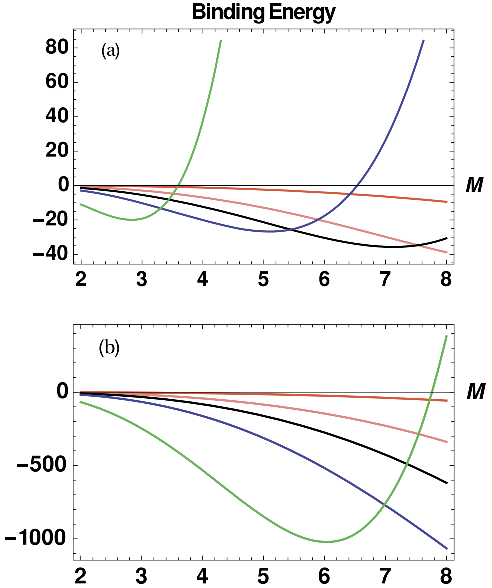

Fig. 1. Binding energy for the quasi-2D NLS model. Figure shows E_b (22) with increasing particle number M for different values of the coupling constant $\alpha = 0.1$ (red), 0.5 (pink), 0.7 (black), 1.0 (blue), 2.0 (green), with parameter μ_0 fixed at (a) 1.0 and (b) 2.4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

NLS model, stabilizes the bound-state, while the counter term E^+ has a destabilizing effect. The graphs show clearly that the stability region of the bound-state for our model shrinks with the increase of the particle number M as well as with the coupling constant α, which is rather anti-intuitive, since for the known NLS model, as seen from E_b^{nls}, the bound-state stability always increases with increasing M and α. On the other hand, one can enlarge the stability domain in our model by increasing the particle momentum μ_0, as evident by comparing the figures (a) and (b). This feature however cannot be matched with the known result of the 1D NLS model, since its binding-energy is independent of μ_0. This shows the intricate nature of the bound-state configuration for our model in comparison with known result of the 1D NLS model.

5. Conclusion and outlook

Going beyond the known form of the existing integrable quantum models in 1D, we propose an alternative higher order Lax matrix approach, exploiting the concept of multi-space–time dimension hidden in integrable systems, and apply it for constructing and solving a novel quasi-2D quantum NLS field model. The key to our success in proving the crucial quantum Yang–Baxter equation, which guarantees the quantum integrability of the model, is the discovery of a new type of field operator algebra, not covered by the existing rules.

The known 1D quantum integrable models satisfying QYBE with rational R-matrix may be realized from an ancestor Lax matrix associated with a spin-like algebra, reducible to conventional spin, bosonic, or the canonical algebra, related to the existing models. A bosonic realization of this matrix was proposed as a general L-operator in Ch. VIII.4 of [3], followed by a theorem, stating that the same L-operator can construct a monodromy matrix with arbitrary rational function.

However this theorem, claiming only a sufficient but not a necessary condition, does not rule out the possibility of alternative L-operators, an example of which is provided by the present Lax matrix.

We stress that, the quantum Lax operator (9) with higher scaling dimension and the associated novel commutation relations (10)–(11) for the fields in our model are fundamentally different from those used in nonrelativistic integrable quantum systems. Unlike the known L-operators belonging to the rational class, (9) with λ^2 spectral dependence and having x-derivative of the field, cannot be realized straightforwardly from any linear combination of the general ultralocal L-operators proposed earlier [3,24]. Similarly, the crucial algebraic structure (10) (with x-derivative term) is different from known ultralocal algebras and evidently cannot be generated by combining them. However, a possible construction of such higher-order Lax operators as a nonlinear combination (like product) of lower order Lax operators of [3,24] and obtaining the underlying novel algebras from the known ones could be taken up as a challenging future problem.

The dimensionality of the present model with its field $q(x, y, t)$ needs special focus. In one hand, the system shares effectively one-dimensional properties, since it is linked to the 1D NLS hierarchy. This is also reflected in the energy spectrum of the present model, which is similar to that of the higher Hamiltonian in the known NLS hierarchy. On the other hand, the model Hamiltonian (14) contains derivatives of the field q_x, q_y in both x and y variables and similarly both these variables are involved in commutators (10)–(11) as well as in the present Lax matrix, defining the model in quasi-2-dimensional form. Moreover, these 2D structures cannot be reduced to 1D by ignoring the dependence on the other variable.

Due to quantum integrability of our quasi-2D NLS model, the eigenvalue problem can be solved exactly for the commuting set of all its conserved operators, with intriguing result for the many particle scattering and bound states.

It is worth adding that, recently we have constructed a novel quasi-2D quantum Landau–Lifshits model belonging also to the rational class (to be reported elsewhere). It is reasonable to assume therefore, that such quasi-2D quantum models generated by higher-order Lax operators, are not limited only to the present NLS case, but constitute a novel family of quantum integrable systems within the rational class. The present approach, general enough for applying to other quasi-higher dimensional quantum models, could open up a new direction in the theory of quantum integrable systems. It is a challenge to find a possible q-deformation of the algebra found here, which could lead to a novel class of quantum algebra, while an exact lattice version of the present Lax matrix could unravel a higher-order ancestor Lax operator for generating a new family of integrable quantum models.

Appendix A

In QYBE (1) with R-matrix (5) and discretized version U^j of the quantum Lax matrix (9), out of total 16 matrix operator relations, except 4 diagonal and 2 extreme off-diagonal terms, all other 10 relations Q^{ij}_{kl} stand nontrivial and their validity needs to be proved using the CR, discretized from (10)–(11):

$$[q_j, q^\dagger_{j,x}] = -2i\frac{\alpha}{\Delta}, \qquad [q_j, q^\dagger_k] = 0 \tag{23}$$

and their conjugates.

A.1. QYBE relation for matrix elements

Using expressions for $a(\lambda-\mu)$, $b(\lambda-\mu)$, c of (5) and CR (23) we may check the validity of

$$\begin{aligned} Q_{12}^{11} &= a\, U^j{}_{11}(\lambda) U^j{}_{12}(\mu) - b\, U^j{}_{12}(\mu) U^j{}_{11}(\lambda) - c\, U^j{}_{11}(\mu) U^j{}_{12}(\lambda)) \\ &= i\Delta(\lambda-\mu)\, q(-\Delta[q_j^\dagger, q_{j,x}] + 2c) + O(\Delta^2) = 0, \end{aligned} \tag{24}$$

up to order $O(\Delta^2)$. Similarly one proves the conjugate relations Q_{21}^{11}, Q_{11}^{21}, Q_{11}^{12} and similar relations Q_{12}^{22}, Q_{21}^{22}, Q_{22}^{12}, Q_{22}^{21}.

The validity of the remaining two relations can also be proved with the use of the same CR (23):

$$\begin{aligned} Q_{21}^{12} &= b\,[U^j{}_{12}(\lambda), U^j{}_{21}(\mu)] + c\,(U^j{}_{22}(\lambda) U^j{}_{11}(\mu) - U^j{}_{11}(\lambda) U^j{}_{22}(\mu)) \\ &= 2i\Delta^2(\lambda-\mu)(\mu[q_{j,x}, q_j^\dagger] + \lambda[q_{j,x}^\dagger, q_j]) + 4i\Delta c(\mu^2-\lambda^2) = 0, \end{aligned} \tag{25}$$

which is valid exactly in all orders of Δ and similarly for the conjugate relation Q_{12}^{21}. This proves thus the validity of all QYBE relations for our quantum quasi-2D NLS field model, associated with the higher Lax operator (9) and algebraic relations (10)–(11), obtained at the limit $\Delta \to 0$.

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